COMPRESSED SENSING AND RELATED OPTIMIZATIONS

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OUTLINE

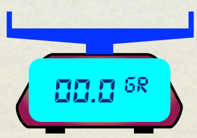
- CS overview
- Preliminaries
- Sparse recovery
- Optimization techniques
- Mixed-norm problems

DIGITAL SCALE EXAMPLE

- 8 coins, one is fake

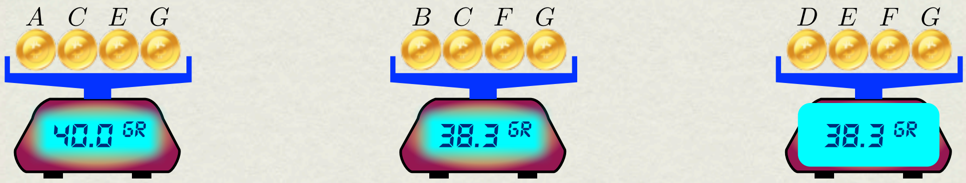
- Find the fake with a digital balance

$= 10^{\text{gr}}$



DIGITAL SCALE EXAMPLE

- 8 coins, one is fake



- None of the coins in experiment 1 is fake.
- All coins not used in experiment 2 are authentic.
- All coins not used in experiment 3 are authentic.

F

$= 8.3^{\text{gr}}$

DIGITAL SCALE EXAMPLE

- Mathematical Formulation

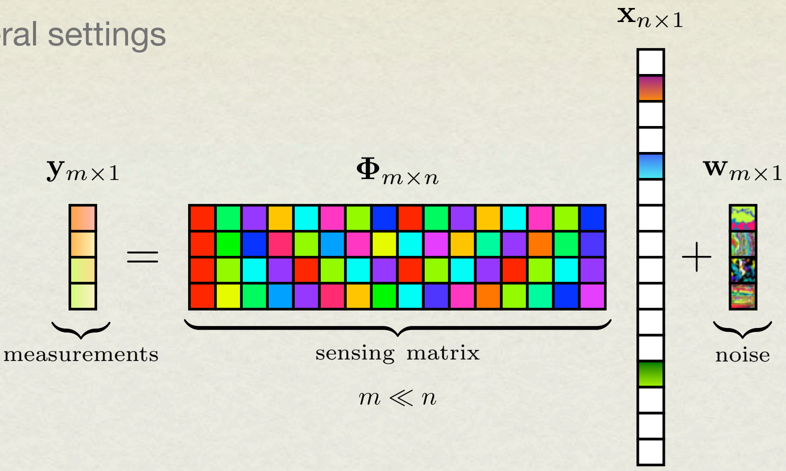
$$\begin{bmatrix} 40 \\ 38.3 \\ 38.3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}}_{\boldsymbol{\Phi}_{3\times 8}} \begin{bmatrix} w_A \\ w_B \\ w_C \\ w_D \\ w_E \\ w_F \\ w_G \\ w_H \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 0 \\ -1.7 \\ -1.7 \end{bmatrix}}_{\mathbf{y}_{3\times 1}} = \boldsymbol{\Phi}_{3\times 8} \underbrace{\begin{bmatrix} w_A - 10 \\ w_B - 10 \\ w_C - 10 \\ w_D - 10 \\ w_E - 10 \\ w_F - 10 \\ w_G - 10 \\ w_H - 10 \end{bmatrix}}_{\mathbf{x}_{8\times 1}} \quad \Rightarrow \quad \mathbf{y}_{3\times 1} = \boldsymbol{\Phi}_{3\times 8} \cdot \mathbf{x}_{8\times 1}$$

- Underdetermined LSoE!
- $\mathbf{x}_{8\times 1}$ = 1-sparse !

CS PROBLEM

- More general settings



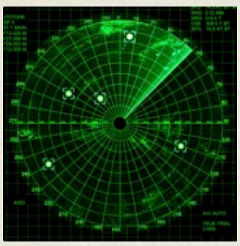
$$\begin{cases} \mathbf{y}_{m\times 1} = \mathbf{\Phi}_{m\times n} \cdot \mathbf{x}_{n\times 1} + \mathbf{w}_{m\times 1} \\ \mathbf{x}_{n\times 1} = k\text{-sparse} \end{cases} \quad \Rightarrow \quad \mathbf{x}_{n\times 1} = ?$$

$m < n \quad \Rightarrow \quad$ infinitely many solutions

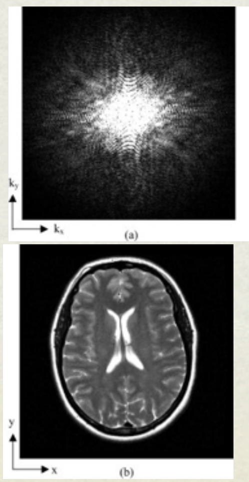
CS PROBLEM

- Similar problems

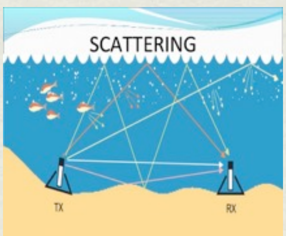
DOA estimation



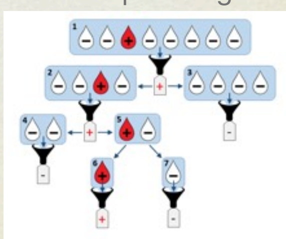
MRI scanning



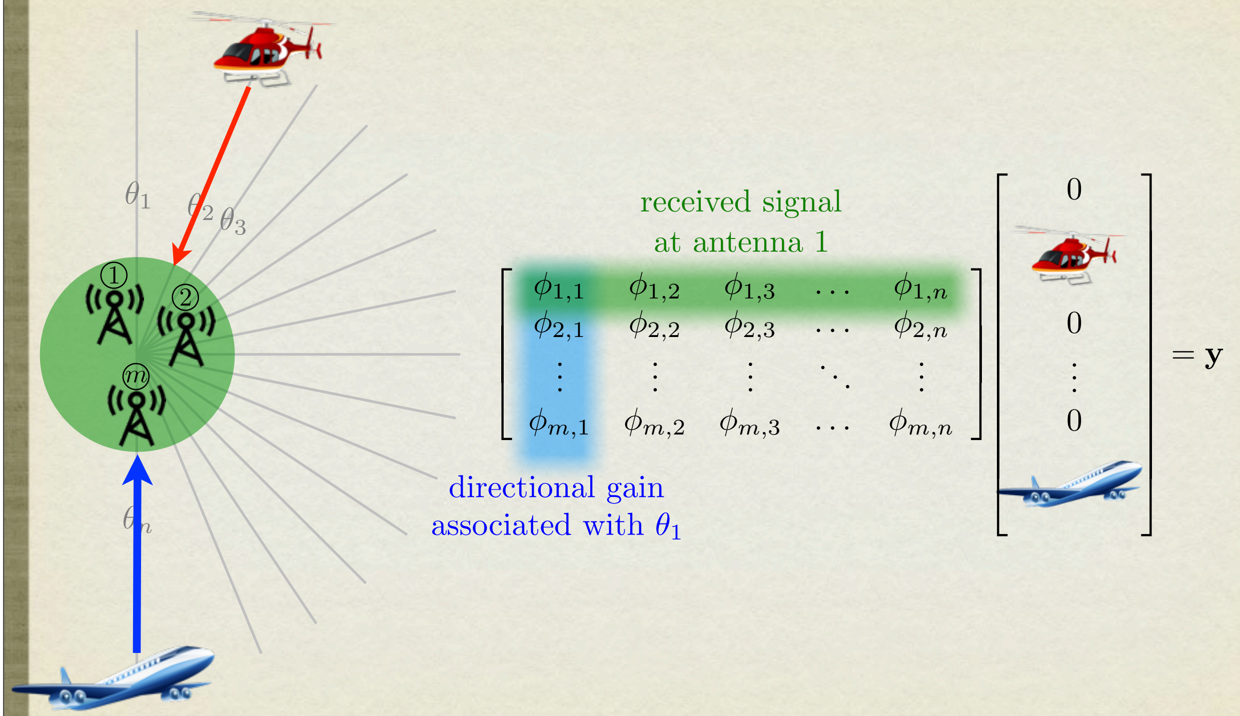
Channel estimation



Group testing



RADAR EXAMPLE



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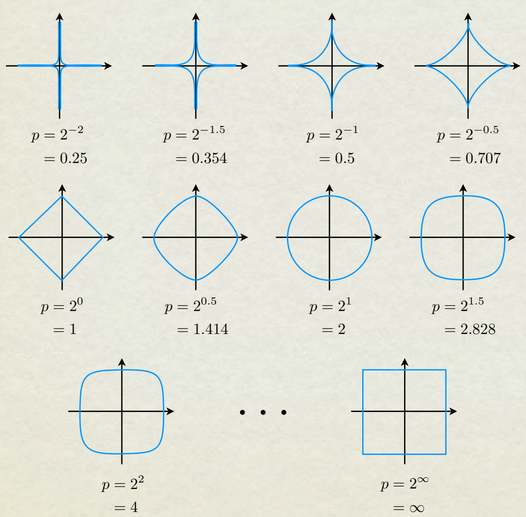
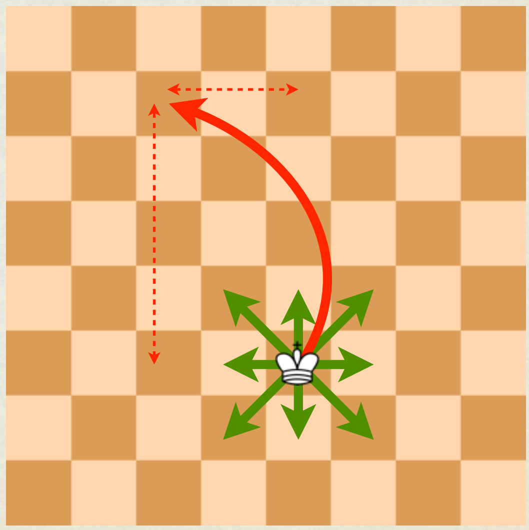
ℓ_p-NORMS

- Unit ball

$$\|\mathbf{x}_{n\times 1}\|_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$$

$$B_1(\mathbb{F}^n; \ell_p) \triangleq \left\{\mathbf{x} \in \mathbb{F}^n \;\middle|\; \|\mathbf{x}\|_p = 1\right\}$$





SUITABLE SAMPLING

$$\mathrm{SPARK}(\boldsymbol{\Phi}_{m\times n}) \triangleq \min \left\{ t \;\middle|\; \left\{ \begin{array}{l} \exists\, i_1, \ldots, i_t \in \{1, \ldots, n\} \\ \exists\, 0 \neq c_1, \ldots, c_t \end{array} \right. \quad : \quad \sum_{j=1}^{t} c_j \boldsymbol{\varphi}_{i_j} = \mathbf{0}_{m\times 1} \right\}$$

$\boldsymbol{\Phi}_{m\times n}$ satisfies $\mathrm{NSP}(k, c_{\mathrm{NSP}})$ iff

$$\forall\, \mathbf{h} \in \mathcal{N}(\boldsymbol{\Phi}),\ \forall\, \Lambda \subset \{1, \ldots, n\},\ |\Lambda| = k : \quad \|\mathbf{h}_\Lambda\|_2 < \frac{c_{\mathrm{NSP}}}{\sqrt{k}} \|\mathbf{h}_{\Lambda^c}\|_1$$

$\boldsymbol{\Phi}_{m\times n}$ satisfies $\mathrm{RIP}(k, \delta_k)$ iff

$$0 \leq \delta_k < 1,\quad \forall\, \mathbf{x}_{n\times 1} \in \Sigma_k : \quad (1-\delta_k)\|\mathbf{x}\|_2^2 \leq \|\boldsymbol{\Phi}_{m\times n}\, \mathbf{x}_{n\times 1}\|_2^2 \leq (1+\delta_k)\|\mathbf{x}\|_2^2$$

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SPARSE RECOVERY

- How to recover **x**?

$$\mathbf{x}_{n\times 1} \in \Sigma_k \quad , \quad \mathbf{y}_{m\times 1} = \mathbf{\Phi}_{m\times n}\, \mathbf{x}_{n\times 1}$$

$$\mathbf{y}_{m\times 1} \overset{?}{\Rightarrow} \mathbf{x}_{n\times 1}$$

$\mathbb{P}_0$

$$\widehat{\mathbf{x}}_{n\times 1} = \text{the sparsest } \mathbf{z}_{n\times 1} \quad \text{s.t.} \quad \mathbf{y}_{m\times 1} = \mathbf{\Phi}_{m\times n}\, \mathbf{z}_{n\times 1}$$

$$\equiv$$

$$\widehat{\mathbf{x}}_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\operatorname{argmin}} \|\mathbf{z}_{n\times 1}\|_0 \quad \text{s.t.} \quad \mathbf{y}_{m\times 1} = \mathbf{\Phi}_{m\times n}\, \mathbf{z}_{n\times 1}$$

- $\text{SPARK}(\mathbf{\Phi}) > 2k \;\Rightarrow\; \mathbf{x}_{n\times 1}$ is the sparsest solution to $\mathbb{P}_0 \;\Rightarrow\; \widehat{\mathbf{x}}_{n\times 1} = \mathbf{x}_{n\times 1}$
- $\mathbb{P}_0$ is non-convex $\Rightarrow$ NP-hard
- $\|\cdot\|_0$ is sensitive to noise

SPARSE RECOVERY

- How to recover **x**?

$$\mathbf{x}_{n\times 1} \in \Sigma_k \quad , \quad \mathbf{y}_{m\times 1} = \mathbf{\Phi}_{m\times n}\, \mathbf{x}_{n\times 1}$$

$$\mathbf{y}_{m\times 1} \overset{?}{\Rightarrow} \mathbf{x}_{n\times 1}$$

- Convex relaxation

$\mathbb{P}_1$

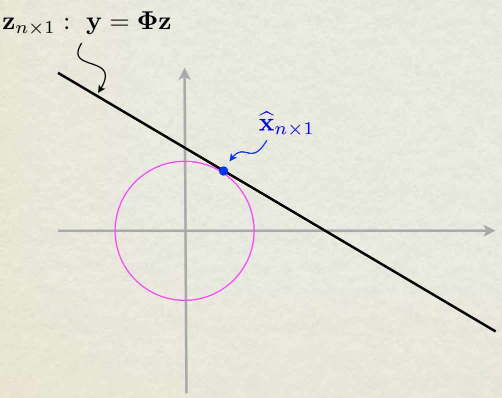
$$\widehat{\mathbf{x}}_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\text{argmin}} \, \|\mathbf{z}_{n\times 1}\|_1 \quad \text{s.t.} \quad \mathbf{y}_{m\times 1} = \mathbf{\Phi}_{m\times n}\, \mathbf{z}_{n\times 1}$$

compressibility instead of sparsity

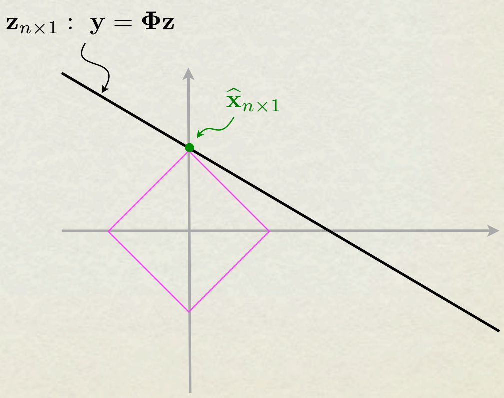
- $\mathbb{P}_1$ is convex $\Rightarrow$ polynomial-time algorithm
- $\mathbb{P}_1$ does not necessarily result in a sparse solution
- $\mathbf{\Phi}$: $\text{NSP}(k,1)$ $\Rightarrow$ $\mathbf{x}_{n\times 1}$ is the unique minimizer of $\mathbb{P}_1$ $\Rightarrow$ $\widehat{\mathbf{x}}_{n\times 1} = \mathbf{x}_{n\times 1}$

GEOMETRICAL PERSPECTIVE

$$\widehat{\mathbf{x}}_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\text{argmin}} \, \|\mathbf{z}_{n\times 1}\|_2 \quad \text{s.t.} \quad \mathbf{y}_{m\times 1} = \mathbf{\Phi}_{m\times n}\, \mathbf{z}_{n\times 1}$$



$$\widehat{\mathbf{x}}_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\text{argmin}} \, \|\mathbf{z}_{n\times 1}\|_1 \quad \text{s.t.} \quad \mathbf{y}_{m\times 1} = \mathbf{\Phi}_{m\times n}\, \mathbf{z}_{n\times 1}$$



$\mathbb{P}_1$ VIA LP

$\mathbb{P}_1$

$$\widehat{\mathbf{x}}_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\text{argmin}} \|\mathbf{z}_{n\times 1}\|_1 \quad \text{s.t.} \quad \mathbf{y}_{m\times 1} = \boldsymbol{\Phi}_{m\times n}\,\mathbf{z}_{n\times 1}$$

LP

$$\widehat{\mathbf{x}}_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\text{argmax}} \left(\mathbf{c}_{n\times 1}\right)^T \mathbf{z}_{n\times 1} \quad \text{s.t.} \begin{cases} \mathbf{z}_{n\times 1} \geq \mathbf{0}_{n\times 1} \\ \mathbf{A}_{m\times n}\,\mathbf{z}_{n\times 1} \leq \mathbf{b}_{m\times 1} \end{cases}$$

$\mathbb{P}_1$ via LP

$$\widehat{\mathbf{x}^{\pm}}_{2n\times 1} = \underset{\mathbf{z}_{2n\times 1}}{\text{argmax}} \; [1\,,\,1\,,\ldots\,,\,1]_{1\times 2n}\,\mathbf{z}_{2n\times 1} \quad \text{s.t.} \begin{cases} \mathbf{z}_{2n\times 1} \geq \mathbf{0}_{n\times 1} \\ \begin{bmatrix} \boldsymbol{\Phi} & -\boldsymbol{\Phi} \\ -\boldsymbol{\Phi} & \boldsymbol{\Phi} \end{bmatrix} \mathbf{z}_{2n\times 1} \leq \begin{bmatrix} \mathbf{y} \\ -\mathbf{y} \end{bmatrix} \end{cases}$$

$$\widehat{x}_i = \left(\widehat{x^{\pm}}\right)_i - \left(\widehat{x^{\pm}}\right)_{n+i}$$

$$1 \leq i \leq n$$

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LP SHORTCOMING

- Basis Pursuit (BP)

$$\mathbf{y}_{m\times n} = \mathbf{\Phi}_{m\times n}\,\mathbf{x}_{n\times 1} \quad \Rightarrow \quad \mathbb{P}_1: \quad \widehat{\mathbf{x}}_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\text{argmin}} \, \|\mathbf{z}_{n\times 1}\|_1 \quad \text{s.t.} \quad \mathbf{y}_{m\times 1} = \mathbf{\Phi}_{m\times n}\,\mathbf{z}_{n\times 1}$$

- Basis Pursuit DeNoising (BPDN)

$$\mathbf{y}_{m\times n} = \mathbf{\Phi}_{m\times n}\,\mathbf{x}_{n\times 1} + \mathbf{w}_{m\times 1}, \quad \|\mathbf{w}_{m\times 1}\|_2 \leq \epsilon \quad \Rightarrow \quad \widetilde{\mathbb{P}}_1: \quad \widehat{\mathbf{x}}_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\text{argmin}} \, \|\mathbf{z}_{n\times 1}\|_1 \quad \text{s.t.} \quad \|\mathbf{y}_{m\times 1} - \mathbf{\Phi}_{m\times n}\,\mathbf{z}_{n\times 1}\|_2 \leq \epsilon$$

- Dantzig Selector

$$\mathbf{y}_{m\times n} = \mathbf{\Phi}_{m\times n}\,\mathbf{x}_{n\times 1} + \mathbf{w}_{m\times 1}, \quad \|\mathbf{\Phi}^T_{m\times n}\mathbf{w}_{m\times 1}\|_\infty \leq \omega \quad \Rightarrow \quad \text{DS}: \quad \widehat{\mathbf{x}}_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\text{argmin}} \, \|\mathbf{z}_{n\times 1}\|_1 \quad \text{s.t.} \quad \|\mathbf{\Phi}^T(\mathbf{y} - \mathbf{\Phi}\,\mathbf{z})\|_\infty \leq \omega$$

LAGRANGE FORM

- BP / BPDN

$\mathbb{P}_1$

$$\hat{\mathbf{x}}_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\text{argmin}} \|\mathbf{z}_{n\times 1}\|_1 \quad \text{s.t.} \quad \mathbf{y}_{m\times 1} = \mathbf{\Phi}_{m\times n}\,\mathbf{z}_{n\times 1}$$

$\widetilde{\mathbb{P}}_1$

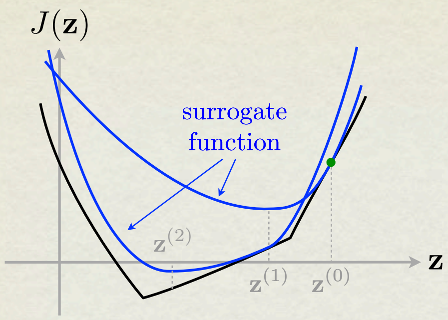
$$\hat{\mathbf{x}}_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\text{argmin}} \|\mathbf{z}_{n\times 1}\|_1 \quad \text{s.t.} \quad \left\|\mathbf{y}_{m\times 1} - \mathbf{\Phi}_{m\times n}\,\mathbf{z}_{n\times 1}\right\|_2 \le \epsilon$$

- Lagrangian form

$$\hat{\mathbf{x}}_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\text{argmin}}\ J(\mathbf{z}_{n\times 1}) = \underset{\mathbf{z}_{n\times 1}}{\text{argmin}}\ \left\|\mathbf{y}_{m\times 1} - \mathbf{\Phi}_{m\times n}\,\mathbf{z}_{n\times 1}\right\|_2^2 + \lambda\,\|\mathbf{z}_{n\times 1}\|_1$$

- λ should be properly set.
- $J(\mathbf{z}_{n\times 1})$ is convex.
- $J(\mathbf{z}_{n\times 1})$ is non-smooth.

SURROGATE FUNCTIONS



- Surrogate function $\widetilde{J}(\mathbf{z}_{n\times 1})$:

   i) $\forall\, \mathbf{z}_{n\times 1}:\ \widetilde{J}(\mathbf{z}) \ge J(\mathbf{z})$.

   ii) For a given $\mathbf{z}^*$: $\widetilde{J}(\mathbf{z}^*) = J(\mathbf{z}^*)$.

   iii) Minimizing $\widetilde{J}$ is easy.

$$\begin{cases} J(\mathbf{z}^{(0)}) & = & \widetilde{J}(\mathbf{z}^{(0)}) \\ \widetilde{J}(\mathbf{z}^{(1)}) & \le & \widetilde{J}(\mathbf{z}^{(0)}) \\ J(\mathbf{z}^{(1)}) & \le & \widetilde{J}(\mathbf{z}^{(1)}) \end{cases} \quad \Rightarrow \quad J(\mathbf{z}^{(1)}) \le J(\mathbf{z}^{(0)}) \quad \Rightarrow \quad \cdots$$

IRLS

$$J(\mathbf{z}_{n\times 1}) = \left\|\mathbf{y}_{m\times 1} - \mathbf{\Phi}_{m\times n}\,\mathbf{z}_{n\times 1}\right\|_2^2 + \lambda\left\|\mathbf{z}_{n\times 1}\right\|_1$$

useful inequality: $\forall\, a, b,\ \ |a| \le \frac{1}{2}\frac{a^2}{|b|} + \frac{1}{2}|b|$

equality for $a = b$

previous iteration

$$\widetilde{J}_{i+1}(\mathbf{z}_{n\times 1}) = \left\|\mathbf{y}_{m\times 1} - \mathbf{\Phi}_{m\times n}\,\mathbf{z}_{n\times 1}\right\|_2^2 + \frac{\lambda}{2}\sum_{j=1}^{n}\left(\frac{z_j^2}{|z_j^{(i)}|} + |z_j^{(i)}|\right)$$

IRLS:

- initialize $\mathbf{z}^{(0)}$
- set $i = 0$
- while (stopping condition)
  - $\mathbf{z}^{(i+1)} = \left(\mathbf{\Phi}^H\mathbf{\Phi} + \frac{\lambda}{2}\,\mathrm{diag}\left(\frac{1}{|z_1^{(i)}|+\epsilon}, \cdots, \frac{1}{|z_n^{(i)}|+\epsilon}\right)\right)^{-1}\mathbf{\Phi}^H\mathbf{y}$
  - $i \leftarrow i + 1$
  - update stop. cond.
- end

ISTA

$$J(\mathbf{z}_{n\times 1}) = \left\|\mathbf{y}_{m\times 1} - \mathbf{\Phi}_{m\times n}\,\mathbf{z}_{n\times 1}\right\|_2^2 + \lambda\left\|\mathbf{z}_{n\times 1}\right\|_1$$

useful inequality: $\forall\,\mathbf{v}_{n\times 1},\ \ \left\|\mathbf{\Phi}_{m\times n}\,\mathbf{v}_{n\times 1}\right\|_2^2 \le \underbrace{\lambda_{\max}\left(\mathbf{\Phi}^H\mathbf{\Phi}\right)}_{L}\|\mathbf{v}\|_2^2$

previous iteration

$$\widetilde{J}_{i+1}(\mathbf{z}_{n\times 1}) = \left\|\mathbf{y} - \mathbf{\Phi}\,\mathbf{z}\right\|_2^2 + L\left\|\mathbf{z} - \mathbf{z}^{(i)}\right\|_2^2 - \left\|\mathbf{\Phi}\left(\mathbf{z} - \mathbf{z}^{(i)}\right)\right\|_2^2 + \lambda\|\mathbf{z}\|_1$$

$$= L\left\|\mathbf{z} - \left(\mathbf{z}^{(i)} - \tfrac{1}{L}\mathbf{\Phi}^H\left(\mathbf{\Phi}\mathbf{z}^{(i)} - \mathbf{y}\right)\right)\right\|_2^2 + \lambda\|\mathbf{z}\|_1 + \left(\begin{array}{c}\text{terms not}\\ \text{having } \mathbf{z}\end{array}\right)$$

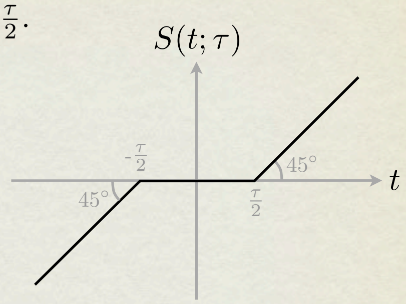
$$\underset{\mathbf{z}_{n\times 1}}{\mathrm{argmin}}\ \widetilde{J}_{i+1}(\mathbf{z}) \quad\Rightarrow\quad \underset{z_j}{\mathrm{argmin}}\ \left|z_j - \left(\mathbf{z}^{(i)} - \tfrac{1}{L}\mathbf{\Phi}^H\left(\mathbf{\Phi}\mathbf{z}^{(i)} - \mathbf{y}\right)\right)_j\right|^2 + \frac{\lambda}{L}|z_j|$$

$\mathbb{F}^n$ minimisation $\ \equiv\ n\,\times\,(\mathbb{F}^1$ minimisation$)$

ISTA

$$S(t;\tau) = \underset{z}{\operatorname{argmin}}\ |z-t|^2 + \tau\,|z| = \begin{cases} \left(|t| - \frac{\tau}{2}\right) e^{j\measuredangle t}, & |t| > \frac{\tau}{2}, \\ 0, & |t| \le \frac{\tau}{2}. \end{cases}$$

shrinkage function



ISTA:

- initialize $\mathbf{z}^{(0)}$
- set $L = \lambda_{\max}(\mathbf{\Phi}^H \mathbf{\Phi})$
- set $i = 0$
- while (stopping condition)
  - $\mathbf{t} \leftarrow \mathbf{z}^{(i)} - \frac{1}{L}\mathbf{\Phi}^H(\mathbf{\Phi}\mathbf{z}^{(i)} - \mathbf{y})$
  - $\mathbf{z}^{(i+1)} = \left[S(t_1; \frac{\lambda}{L}), \ldots, S(t_n; \frac{\lambda}{L})\right]^T$
  - $i \leftarrow i+1$
  - update stop. cond.
- end

ADMM

$$\mathbf{x}^*_{n\times 1} = \underset{\mathbf{z}_{n\times 1}}{\operatorname{argmin}}\ f(\mathbf{z}) + g(\mathbf{z})$$

- ADMM

$$\mathbf{x}^*_{n\times 1} = \underset{\mathbf{z}_{n\times 1}, \boldsymbol{\zeta}_{n\times 1}}{\operatorname{argmin}}\ f(\mathbf{z}) + g(\boldsymbol{\zeta})$$

$$\text{s.t.}\quad \mathbf{z} - \boldsymbol{\zeta} = 0$$

$$J_k(\mathbf{z}, \boldsymbol{\zeta}) \triangleq f(\mathbf{z}) + g(\boldsymbol{\zeta}) + \mu\|\mathbf{z} - \boldsymbol{\zeta}\|_2^2 - (\boldsymbol{\tau}_k)^T(\mathbf{z} - \boldsymbol{\zeta})$$

$$\begin{cases} \boldsymbol{\tau}_{k+1} = \boldsymbol{\tau}_k - 2\mu\left(\mathbf{z}^{(k)} - \boldsymbol{\zeta}^{(k)}\right) \\ \mathbf{z}^{(k+1)} = \operatorname{argmin}_{\mathbf{z}_{n\times 1}}\ J_{k+1}\left(\mathbf{z}, \boldsymbol{\zeta}^{(k)}\right) \\ \boldsymbol{\zeta}^{(k+1)} = \operatorname{argmin}_{\boldsymbol{\zeta}_{n\times 1}}\ J_{k+1}\left(\mathbf{z}^{(k+1)}, \boldsymbol{\zeta}\right) \end{cases}$$

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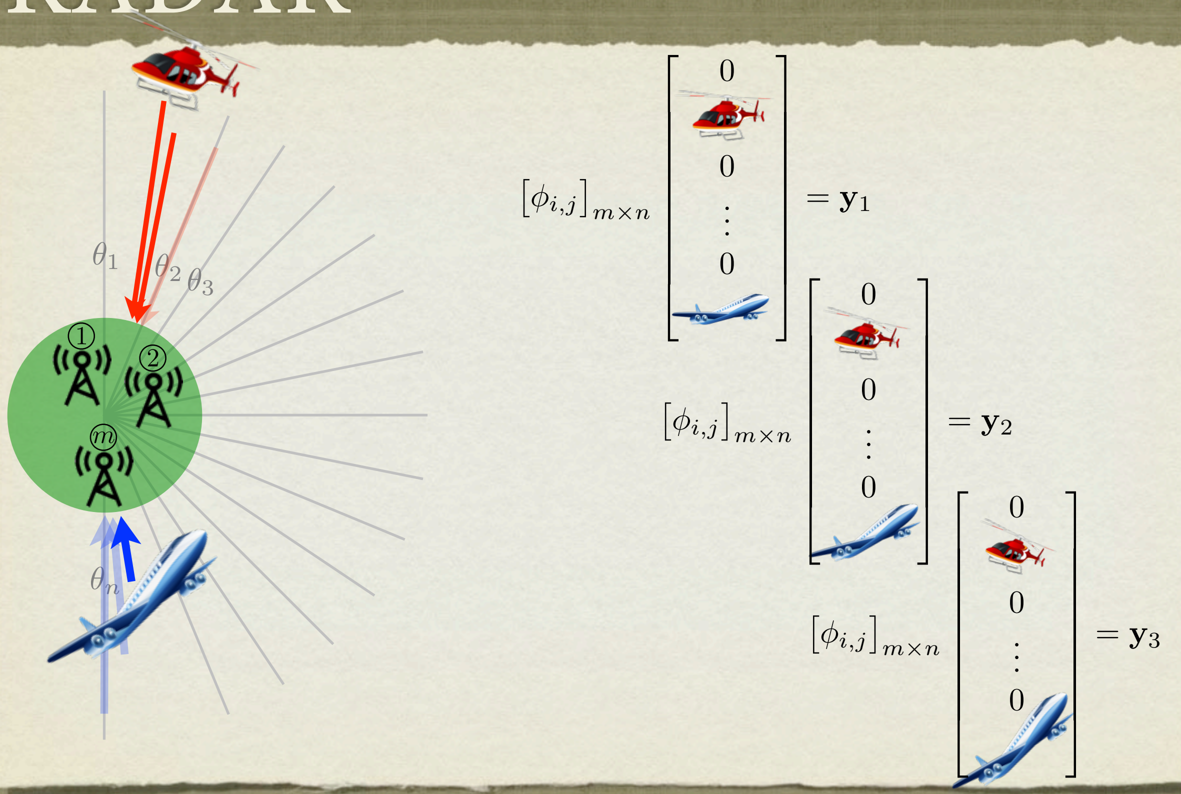
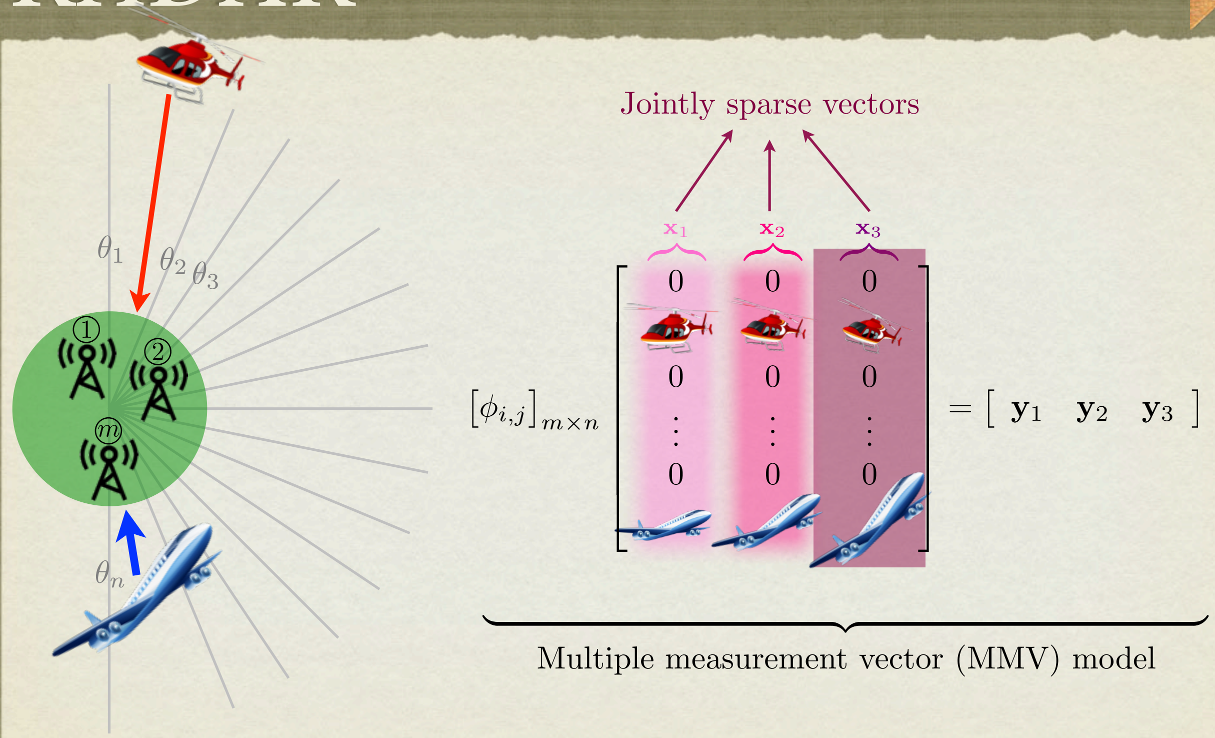
MULTI-SNAPSHOT RADAR



MULTI-SNAPSHOT RADAR



MMV

$$\begin{cases} \underbrace{\mathbf{Y}_{m\times d}}_{\text{measurements}} = \underbrace{\mathbf{\Phi}_{m\times n}}_{\text{sensing matrix}} \underbrace{\mathbf{X}_{n\times d}}_{\text{jointly sparse}} \\ \mathbf{X}_{n\times d} \text{ has few non-zero rows} \end{cases}$$

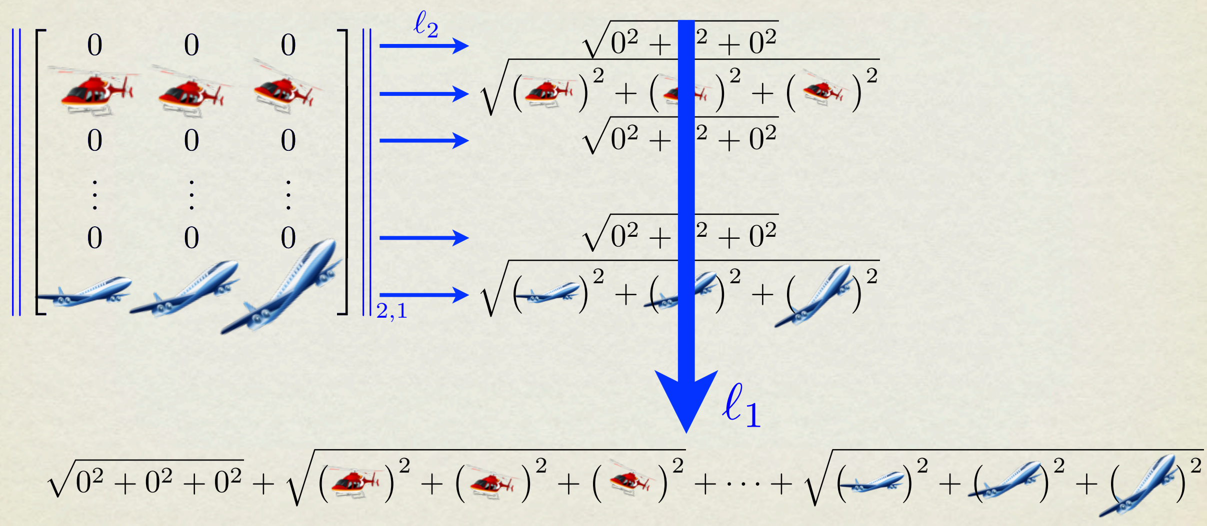
or

$$\begin{cases} \underbrace{\mathbf{Y}_{m\times d}}_{\text{measurements}} = \underbrace{\mathbf{\Phi}_{m\times n}}_{\text{sensing matrix}} \underbrace{\mathbf{X}_{n\times d}}_{\text{jointly sparse}} + \underbrace{\mathbf{W}_{m\times d}}_{\text{noise}} \\ \mathbf{X}_{n\times d} \text{ has few non-zero rows} \end{cases}$$

$$\widehat{\mathbf{X}}_{n\times d} = \underset{\mathbf{Z}_{n\times d}}{\operatorname{argmin}} \|\mathbf{Z}_{n\times d}\|_{2,1} \quad \text{s.t.} \quad \mathbf{Y}_{m\times d} = \mathbf{\Phi}_{m\times n}\mathbf{Z}_{n\times d}$$

$$\widehat{\mathbf{X}}_{n\times d} = \underset{\mathbf{Z}_{n\times d}}{\operatorname{argmin}} \|\mathbf{Z}_{n\times d}\|_{2,1} \quad \text{s.t.} \quad \|\mathbf{Y}_{m\times d} - \mathbf{\Phi}_{m\times n}\mathbf{Z}_{n\times d}\|_2 \le \epsilon$$

$\ell_{2,1}$



$\ell_{2,1}$

$$\widehat{\mathbf{X}}_{n\times d} = \underset{\mathbf{Z}_{n\times d}}{\operatorname{argmin}} \|\mathbf{Z}_{n\times d}\|_{2,1} \quad \text{s.t.} \quad \mathbf{Y}_{m\times d} = \mathbf{\Phi}_{m\times n}\,\mathbf{Z}_{n\times d}$$

Second order cone programming (SOCP)

$$\underset{t_1,\ldots,t_n,\mathbf{Z}_{n\times d}}{\operatorname{argmin}} \sum_{i=1}^{n} t_i$$

$$\text{s.t.} \begin{cases} 0 \le t_i & , \quad \forall 1 \le i \le n \\ \left\| [Z_{i,1}\,,\, Z_{i,2}\,\ldots\, Z_{i,d}] \right\|_2 \le t_i & , \quad \forall 1 \le i \le n \\ \mathbf{Y}_{m\times d} = \mathbf{\Phi}_{m\times n}\,\mathbf{Z}_{n\times d} \end{cases}$$

Lagrange form + ADMM

$$\underset{\mathbf{Z}_{n\times d},\mathbf{B}_{m\times d},\mathbf{C}_{m\times d}}{\operatorname{argmin}} \|\mathbf{Y} - \underbrace{\mathbf{B}}_{\mathbf{\Phi}\mathbf{Z}}\|_2^2 + \lambda\|\mathbf{Z}\|_{2,1} + \mu\|\mathbf{B} - \mathbf{\Phi}\mathbf{Z}\|_2^2 - \operatorname{tr}\big(\mathbf{C}^T(\mathbf{B} - \mathbf{\Phi}\mathbf{Z})\big)$$

Thank you!

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