On Robust Optimization

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- **Ellipsoid** (Ben-Tal and Nemirovski):

$$\mathcal{U}_0 = \left\{A \in \mathbb{R}^{m\times n} \mid A = A^{(0)} + \sum_{j=1}^{k} u_j A^{(j)},\ \|u\|_2 \leq 1,\ u \in \mathbb{R}^k\right\}.$$

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- $$\begin{aligned} \min \quad & c^t x \\ \text{s.t.} \quad & a^{i(0)} x \leq b_i - \left\| \Delta_i^T x \right\|_2, \quad i = 1, \ldots, m, \end{aligned}$$

  where $\Delta_i = [a^{i(1)^T}, a^{i(2)^T}, \ldots, a^{i(k)^T}]$.

Example 1: Inflation rate.

- Unstable Economics: Robust Optimization;
- Stable Economics: Robustness/Stability;
- A new production line: Sensitivity Analysis.

Example 2: Mathematical optimization in intensity modulated radiation therapy.

Ehrgott et al. Ann Oper Res (2010) 175: 309-365.

استفاده از بهینه‌سازی برای بهبود روش‌های پرتودرمانی در درمان سرطان

حامد اعظمی زنوزق، مسعود زارع‌پیشه، و مجید سلیمانی دامنه

فرهنگ و اندیشهٔ ریاضی
سال ۳۹، شمارهٔ ۶۶ (بهار و تابستان ۱۳۹۹) صص. ۴۱ تا ۷۵

چکیده

مقابله با سرطان یکی از مهمترین چالش‌های بشر است. با وجود پیشرفت‌های روزافزون بشری، هنوز یک درمان قطعی برای این دسته از بیماری‌ها پیدا نشده است. یکی از روش‌های مرسوم برای درمان سرطان، پرتودرمانی است. این روش، برای درمان انواع مختلف سرطان برای حدود یک قرن مورد استفاده قرار گرفته و در ۱۵ سال اخیر بهبود قابل توجهی یافته است. در این نوشتار، برخی مدل‌های بهینه‌سازی مورد استفاده در پرتودرمانی را بررسی می‌کنیم. روش پرتودرمانی با شدت تعدیل‌شده و مدل‌بندی‌های مختلف بهینه‌سازی در آن را شرح می‌دهیم. به‌علاوه، برخی از چالش‌های مهم در این زمینه از جمله محدب نبودن مسئله و چندهدفه بودنِ آن مورد بررسی قرار می‌گیرد و پیشنهادهایی برای غلبه بر این چالش‌ها بیان می‌شود.

Example 3: Machine learning - SVM

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$$h^{w,b}(x) = sgn(\langle w, x\rangle + b) = \begin{cases} -1, & \langle w, x\rangle + b < 0 \\ +1, & \langle w, x\rangle + b > 0 \end{cases}$$

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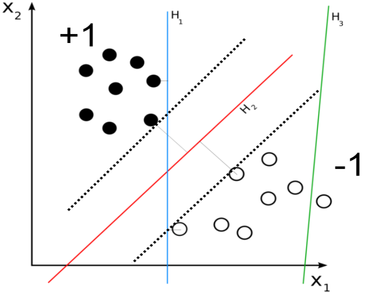

- Soft margin:

$$\min_{w,b,\xi} \quad \frac{1}{2}\|w\|_2^2 + C\sum_{i=1}^{m}\xi_i$$

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- Robust model:

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- T. Trafalis and R. Gilbert. Robust support vector machines for classification and computational issues. Optimization Methods and Software, 22(1):187-198, 2007.

Robustness: Linear case

- Consider the following linear programming (LP) problem:

$$LP(c): \quad \max \; c^t x \quad \text{s.t.} \; Ax \leqq b, \tag{1}$$

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- By LP(c) we denote LP (1) with objective coefficient vector c.
- A feasible solution $x^* \in X$ is called an optimal solution to LP(c) if $c^t x^* \geq c^t x$ for each $x \in X$.

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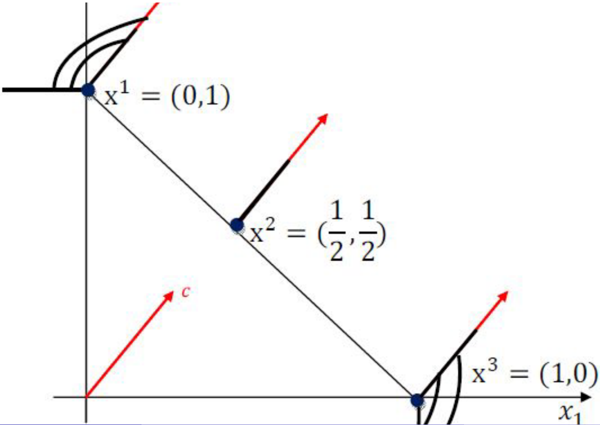
Lemma

$x^* \in X$ is an optimal solution to LP(c) if and only if $c \in \mathcal{A}^*$.

Example:

$$\begin{array}{ll} \max & x_1 + x_2 \\ s.t. & x_1 + x_2 \leq 1, \\ & -x_1 \leq 0, \\ & -x_2 \leq 0. \end{array}$$

$$A_{I(x^1)} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \; A_{I(x^2)} = \begin{pmatrix} 1 & 1 \end{pmatrix} \text{ and } A_{I(x^3)} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$



- We say that $x^0 \in S \subseteq \mathbb{R}^n$ is a relative interior point of degree k for S if there exist scalar $0 < \epsilon \in \mathbb{R}$ and an affine subspace H with $\dim(H) = k$ such that

$$B(x^0;\epsilon) \cap H \subseteq S,$$

where $B(x^0;\epsilon) = \{x \in \mathbb{R}^n : \|x - x^0\| < \epsilon\}$ is the ball centered at x^0 with radius ϵ.

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- If $k = \dim(S)$, then x^0 is called a relative interior point of S. It is clear that, if $k = n$, then x^0 is an interior point of S.
- We denote the set of relative interior points of S by $ri(S)$. Also, the set of relative interior points of degree k for S is denoted by $ri(S; k)$.

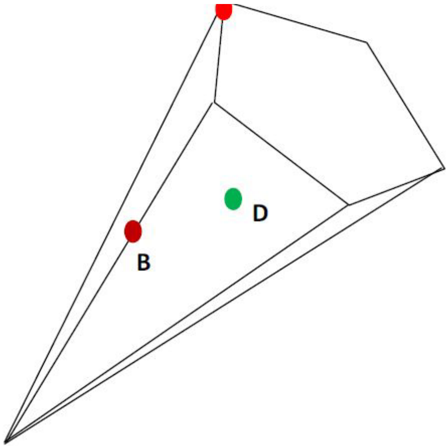

Figure : E, B and D are the relative interior points of orders 0, 1 and 2, respectively.

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- See also Deb and Gupta (2006), Hladík (2008), Hladík and Sitarz (2013), and Georgiev et al. (2013).

Definition

Georgiev et al. (2013) Let x^* be an optimal solution to LP(c). It is said to be robust in the norm sense if there exists scalar $\epsilon > 0$ such that x^* is an optimal solution to LP($c + d$) for each $d \in \mathbb{R}^n$ with $\|d\| \leq \epsilon$.

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Theorem

Let $x^* \in X$ be a feasible solution to Problem (1). Then the following three statements are equivalent:

(i) $c \in int(\mathcal{A}^*)$.

(ii) *There exists scalar $\epsilon > 0$ such that x^* is an optimal solution to $LP(c+d)$ for each $d \in \mathbb{R}^n$ with $\|d\| \leq \epsilon$.*

(iii) *x^* is a unique optimal solution to $LP(c)$.*

Lemma

$x^* \in X$ is an optimal solution to $LP(c)$ if and only if $c \in \mathcal{A}^*$.

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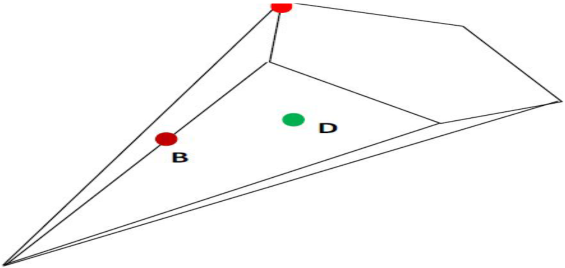
Definition

Let $x^* \in X$ be an optimal solution to LP(c). Then x^* is called a robust optimal solution (robust solution briefly) of order k ($k \leq n$) if $c \in ri(\mathcal{A}^*; k)$, i.e., the largest degree of interiority of c in $\mathcal{A}^*$ is equal to k. The quantity k is called the robustness order of x^* and it is denoted by $RO(x^*)$.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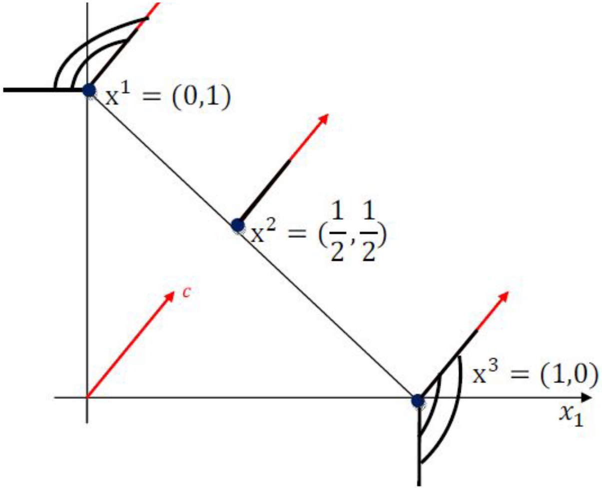

$x^1 = (0,1)$
$x^2 = (\frac{1}{2}, \frac{1}{2})$
c
$x^3 = (1,0)$
x_1

Calculating the robustness order: LP case

Theorem

Let $x^* \in X$ be an optimal solution to Problem (1). Consider the following LP problem

$$\begin{array}{rl} \max & \sum_{i \in I(x^*)} y_i \\ s.t. & \sum_{i \in I(x^*)} (y_i + w_i) a^i = \alpha c^t \\ & w_i \geq 0, \quad 0 \leq y_i \leq 1, \ i \in I(x^*), \\ & \alpha \geq 1. \end{array} \tag{2}$$

Let $(w^*, y^*, \alpha^*) \in \mathbb{R}^{|I(x^*)|} \times \mathbb{R}^{|I(x^*)|} \times \mathbb{R}$ be an optimal solution to Problem (2), and E^* be the submatrix of A whose rows are a^i's with $y_i^* > 0$. Then $RO(x^*) = rank(E^*)$.

Robustness with respect to the angle deviation: LP case

Definition

Let $0 \neq c, d \in \mathbb{R}^n$ and $\theta \in (0, \pi]$. Also, let $x^* \in X$ be an optimal solution to LP(c). Then, θ is said to be an eligible angle deviation of c at x^* in direction d if there exists scalar $\alpha > 0$ such that

i. $c + \alpha d \neq 0$,

ii. $\angle(c, c + \alpha d) \geq \theta$, and

iii. x^* is an optimal solution to LP($c + \alpha d$).

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Definition

Let $x^* \in X$ be an optimal solution to LP(c). Then, $\theta \in (0, \pi]$ is said to be an eligible angle deviation of c at x^* if it is an eligible angle deviation of c at x^* in some direction $d \in \mathbb{R}^n$.

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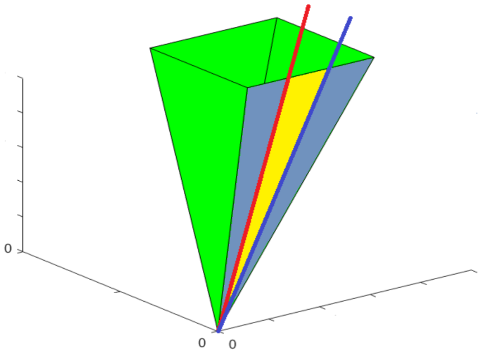
Definition

Let $x^* \in X$ be an optimal solution to LP(c). Then, $\theta \in (0, \pi]$ is said to be an eligible angle deviation of c at x^* if it is an eligible angle deviation of c at x^* in some direction $d \in \mathbb{R}^n$.

If θ is an eligible angle deviation of c at x^*, then θ' is also an eligible angle deviation of c at x^* for each $\theta' \in (0, \theta]$.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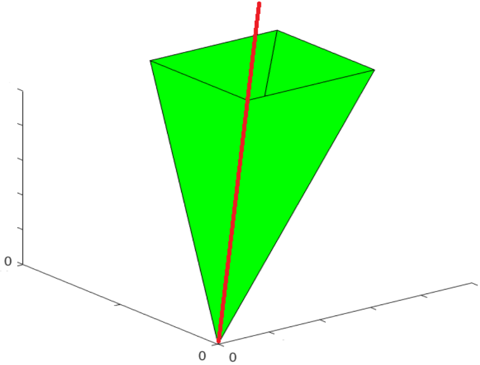
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Theorem

Let $x^* \in X$ be an optimal solution to $LP(c)$, and z^* be the optimal value of the following optimization problem:

$$
\begin{aligned}
z^* = \min \quad & z \\
s.t. \quad & w^t A_{I(x^*)} c \leq z \|w^t A_{I(x^*)}\| \|c\|, \\
& \|w^t A_{I(x^*)}\| \geq 1, \\
& w \geq 0.
\end{aligned}
\tag{3}
$$

Let θ^* denote the largest eligible angle deviation of c at x^*. Then

i) $z^* \in [-1, 1]$.

ii) *If $z^* = 1$, then there is no eligible angle deviation of c at x^*.*

iii) *If $z^* < 1$, then $\theta^* = \arccos(z^*)$. Moreover, $d = A^t_{I(x^*)} w^* - c$ is a direction with the largest eligible angle deviation, where w^* is a part of an optimal solution of Problem (3).*

Robust Solutions: MOLP case

$$MOLP(C): \quad \max \; Cx \quad \text{s.t.} \; Ax \leqq b,$$

where $C_{p\times n}$, $A_{m\times n}$, $b_{m\times 1}$. Also, X is the set of feasible solutions.

Robust Solutions: MOLP case

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Definition

$x^* \in X$ is called an efficient solution to MOLP(C) if there does not exist $x \in X$ such that $Cx \geq Cx^*$.

Robust Solutions: MOLP case

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Definition

$x^* \in X$ is called an efficient solution to MOLP(C) if there does not exist $x \in X$ such that $Cx \geq Cx^*$.

Definition

$x^* \in X$ is called a strictly efficient solution to MOLP(C) if it is an efficient solution and furthermore $\{x \in X \; : \; Cx = Cx^*\} = \{x^*\}$.

Definition

(Georgiev et al. 2013) Let x^* be an efficient solution to MOLP(C). It is said to be a robust efficient solution in the norm sense if there exists scalar $\epsilon > 0$ such that x^* is an efficient solution to MOLP($C + D$) for each $p \times n$ matrix D with $\|D\| \leq \epsilon$.

Definition

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Theorem

$x^* \in X$ is a robust efficient solution of MOLP(C) in the norm sense if and only if it is a strictly efficient solution to MOLP(C).

Lemma

$x^* \in X$ is an efficient solution to MOLP(C) if and only if $ri(\mathfrak{C}) \cap \mathcal{A}^* \neq \emptyset$, where $ri(\mathfrak{C}) = \{\lambda^t C : \ \lambda > 0, \ \lambda \in \mathbb{R}^p\}$ and $\mathcal{A}^* = Pos\{(a^i)^T : a^i x^* = b_i\}$.

Lemma

$x^* \in X$ is an efficient solution to MOLP(C) if and only if $ri(\mathfrak{C}) \cap \mathcal{A}^* \neq \emptyset$, where $ri(\mathfrak{C}) = \{\lambda^t C : \ \lambda > 0, \ \lambda \in \mathbb{R}^p\}$ and $\mathcal{A}^* = Pos\{(a^i)^T : a^i x^* = b_i\}$.

Definition

Let $x^* \in X$ be an efficient solution to MOLP(C). Then x^* is called a robust efficient solution (robust solution briefly) of order k ($k \leq n$) if $ri(\mathfrak{C}) \cap ri(\mathcal{A}^*; k) \neq \emptyset$. The scalar k is called the robustness order of x^* and it is denoted by $RO^{molp}(x^*)$.

Lemma

$x^* \in X$ is an efficient solution to MOLP(C) if and only if $ri(\mathfrak{C}) \cap \mathcal{A}^* \neq \emptyset$, where $ri(\mathfrak{C}) = \{\lambda^t C : \ \lambda > 0, \ \lambda \in \mathbb{R}^p\}$ and $\mathcal{A}^* = Pos\{(a^i)^T : a^i x^* = b_i\}$.

Definition

Let $x^* \in X$ be an efficient solution to MOLP(C). Then x^* is called a robust efficient solution (robust solution briefly) of order k ($k \leq n$) if $ri(\mathfrak{C}) \cap ri(\mathcal{A}^*; k) \neq \emptyset$. The scalar k is called the robustness order of x^* and it is denoted by $RO^{molp}(x^*)$.

Theorem

If x^* is a robust efficient solution of order n, then x^* is robust in the sense of norm.

Theorem

Let $x^* \in X$ be an efficient solution to MOLP (C). Consider the following LP problem

$$\begin{array}{ll} \max & \sum_{i\in I(x^*)} y_i \\ s.t. & \sum_{i\in I(x^*)} (y_i + w_i)a^i = \sum_{i=1}^{p} \alpha_i c^i \\ & w_i \geq 0, \quad 0 \leq y_i \leq 1, \ i \in I(x^*), \\ & \alpha_i \geq 1, \quad i = 1,2,\ldots,p. \end{array} \tag{4}$$

Let $(w^*, y^*, \alpha^*) \in \mathbb{R}^{|I(x^*)|} \times \mathbb{R}^{|I(x^*)|} \times \mathbb{R}^p$ be an optimal solution to Problem (4), and E^* be the submatrix of A whose rows are a^i's with $y_i^* > 0$. Then $RO^{molp}(x^*) = rank(E^*)$.

Definition

Let C, D be two $p \times n$ nonzero matrices. Also, let $\theta \in (0, \pi]$ and let $x^* \in X$ be an efficient solution to MOLP(C). Then, θ is said to be an eligible angle deviation of C at x^* in direction D if there exist some p-vector $\lambda > 0$ and some scalar $\alpha > 0$ such that

i. $\lambda^t(C + \alpha D) \neq 0$,

ii. $\angle(\lambda^t C, \lambda^t(C + \alpha D)) \geq \theta$, and

iii. $\lambda^t C, \lambda^t(C + \alpha D) \in \mathcal{A}^*$.

Definition

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ii. $\angle(\lambda^t C, \lambda^t(C + \alpha D)) \geq \theta$, and
iii. $\lambda^t C, \lambda^t(C + \alpha D) \in \mathcal{A}^*$.

Definition

Let $x^* \in X$ be an efficient solution to MOLP(C). Then, $\theta \in (0, \pi]$ is said to be an eligible angle deviation of C at x^* if it is an eligible angle deviation of C at x^* in direction some nonzero $p \times n$ matrix D.

Theorem

Let $x^* \in X$ be an efficient solution to MOLP(C), and z^* be the optimal value of the following optimization problem:

$$\begin{aligned} z^* = \min \quad & z \\ s.t. \quad & w^t A_{I(x^*)} C^t \lambda \leq z \|w^t A_{I(x^*)}\| \|\lambda^t C\|, \\ & v^t A_{I(x^*)} = \lambda^t C, \\ & \|w^t A_{I(x^*)}\| \geq 1, \\ & v, w, \lambda \geq 0, \\ & e^t \lambda \geq 1. \end{aligned} \tag{5}$$

Let θ^* denote the largest eligible angle deviation of C at x^* and $(v^*, w^*, \lambda^*, z^*)$ be an optimal solution to Problem (5) with $\lambda_1^* > 0$ (without loss of generality). Let D^* be a $p \times n$ matrix whose first row is $\frac{1}{\lambda_1^*}(w^{*t} A_{I(x^*)} - \lambda^t C)$ and its other rows are zero. Then
(i) $z^* \in [-1, 1]$.

Theorem

Let $x^* \in X$ be an efficient solution to MOLP(C), and z^* be the optimal value of the following optimization problem:

$$\begin{aligned} z^* = \min \quad & z \\ s.t. \quad & w^t A_{I(x^*)} C^t \lambda \leq z \|w^t A_{I(x^*)}\| \|\lambda^t C\|, \\ & v^t A_{I(x^*)} = \lambda^t C, \\ & \|w^t A_{I(x^*)}\| \geq 1, \\ & v, w, \lambda \geq 0, \\ & e^t \lambda \geq 1. \end{aligned} \tag{5}$$

Let θ^* denote the largest eligible angle deviation of C at x^* and $(v^*, w^*, \lambda^*, z^*)$ be an optimal solution to Problem (5) with $\lambda_1^* > 0$ (without loss of generality). Let D^* be a $p \times n$ matrix whose first row is $\frac{1}{\lambda_1^*}(w^{*t} A_{I(x^*)} - \lambda^t C)$ and its other rows are zero. Then
(i) $z^* \in [-1, 1]$. *(ii) If $z^* = 1$, then there is no eligible angle deviation of C at x^*.*

Theorem

Let $x^* \in X$ be an efficient solution to MOLP(C), and z^* be the optimal value of the following optimization problem:

$$\begin{array}{rll} z^* = \min & z & \\ s.t. & w^t A_{I(x^*)} C^t \lambda \leq z \|w^t A_{I(x^*)}\| \|\lambda^t C\|, & \\ & v^t A_{I(x^*)} = \lambda^t C, & \\ & \|w^t A_{I(x^*)}\| \geq 1, & (5) \\ & v, w, \lambda \geq 0, & \\ & e^t \lambda \geq 1. & \end{array}$$

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(i) $z^* \in [-1, 1]$. (ii) If $z^* = 1$, then there is no eligible angle deviation of C at x^*. (iii) If $z^* < 1$, then $\theta^* = \arccos(z^*)$. Moreover, D^* is a direction with the largest eligible angle deviation.

Definition

Let C, D be two $p \times n$ nonzero matrices. Also, let $\theta \in (0, \pi]$, let $x^* \in X$ be an efficient solution to MOLP(C) and let $\lambda^* > 0$ be a p-vector. Then, θ is said to be a λ^*-eligible angle deviation of C at x^* in direction D if x^* solves $\max\{\lambda^{*t} Cx : \ x \in X\}$ and there exists some scalar $\alpha > 0$ such that i. $\lambda^{*t}(C + \alpha D) \neq 0$, ii. $\angle(\lambda^{*t} C, \lambda^{*t}(C + \alpha D)) \geq \theta$, and iii. $\lambda^{*t} C, \lambda^{*t}(C + \alpha D) \in \mathcal{A}^*$.

Theorem

Let $\lambda^* > 0$ be a given p-vector. Let x^* be an efficient solution to $MOLP(C)$ and k be a natural number.

Theorem

Let $\lambda^* > 0$ be a given p-vector. Let x^* be an efficient solution to $MOLP(C)$ and k be a natural number. If there exist k matrices $D^1, D^2, \ldots, D^k$ such that $\lambda^{*t} D^1, \lambda^{*t} D^2, \ldots, \lambda^{*t} D^k$ are linearly-independent and there is a positive λ^*-eligible angle deviation of C at x^* in direction $\pm D^i$ for each $i \in \{1, 2, \ldots, k\}$, then $RO^{molp}(x^*) \geq k$. If k is the biggest natural number with this property, then $RO(x^*)^{molp} = k$.

Theorem

Let $\lambda^* > 0$ be a given p-vector. Let x^* be an efficient solution to $MOLP(C)$ and k be a natural number. If there exist k matrices $D^1, D^2, \ldots, D^k$ such that $\lambda^{*t} D^1, \lambda^{*t} D^2, \ldots, \lambda^{*t} D^k$ are linearly-independent and there is a positive λ^*-eligible angle deviation of C at x^* in direction $\pm D^i$ for each $i \in \{1, 2, \ldots, k\}$, then $RO^{molp}(x^*) \geq k$. If k is the biggest natural number with this property, then $RO(x^*)^{molp} = k$. Conversely, if $RO^{molp}(x^*) = k \geq 2$, then there exist a $p-$vector $\lambda^* > 0$ and k matrices $D^1, D^2, \ldots, D^k$ such that $\lambda^{*t} D^1, \lambda^{*t} D^2, \ldots, \lambda^{*t} D^k$ are linearly-independent and there is a positive λ^*-eligible angle deviation of C at x^* in direction $\pm D^i$ for each $i \in \{1, 2, \ldots, k\}$.

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Robustness: Nonlinear case

- Multiple Objective Programming (MOP):

$$(MOP): \qquad Min\{f(x) \ : \ x \in X\},$$

where $f(x) = (f_1(x), f_2(x), \ldots, f_p(x))^T$, where $X \subseteq \mathbb{R}^n$ is a nonempty set; and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$.

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Definition

$x^* \in X$ is called a Pareto (efficient) solution to MOP if $\nexists x^o \in X$ such that

$$f_j(x^o) \leq f_j(x^*) \text{ for each } j = 1, 2, \ldots, p,$$

$$f_j(x^o) < f_j(x^*) \text{ for some } j = 1, 2, \ldots, p.$$

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$$f_j(x^o) < f_j(x^*) \text{ for some } j = 1, 2, \dots, p.$$

Definition

$x^* \in X$ is called a weak Pareto (weak efficient) solution to MOP if $\nexists x^o \in X$ such that $f_j(x^o) < f_j(x^*)$ for each $j = 1, 2, \dots, p$.

Definition

Let $K \subseteq \mathbb{R}^p$ be a nontrivial closed pointed convex cone. $x^* \in X$ is called an efficient solution to MOP w.r.t K if

$$(f(x^*) - K) \cap f(X) = \{f(x^*)\}.$$

Definition

A feasible solution $\hat{x} \in X$ is called properly efficient solution of MOP in the sense of Geoffrion, if it is efficient and there is a real number $M > 0$ such that for all $i \in \{1, 2, ..., p\}$ and $x \in X$ satisfying $f_i(x) < f_i(\hat{x})$ there exists an index $j \in \{1, 2, ..., p\}$ such that $f_j(x) > f_j(\hat{x})$ and

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq M.$$

Definition

A feasible solution $\hat{x} \in X$ is called properly efficient solution of MOP in the sense of Geoffrion, if it is efficient and there is a real number $M > 0$ such that for all $i \in \{1, 2, ..., p\}$ and $x \in X$ satisfying $f_i(x) < f_i(\hat{x})$ there exists an index $j \in \{1, 2, ..., p\}$ such that $f_j(x) > f_j(\hat{x})$ and

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Definition

A feasible solution $\hat{x} \in X$ is called properly efficient solution of MOP in the sense of Henig, if $(f(\hat{x}) - C) \cap f(X) = \{f(\hat{x})\}$, for some convex pointed cone C with $\mathbb{R}^p_{\geqq} \backslash \{0\} \subseteq int(C)$.

Definition

A feasible solution $\hat{x} \in X$ is called properly efficient solution of MOP in the sense of Geoffrion, if it is efficient and there is a real number $M > 0$ such that for all $i \in \{1, 2, ..., p\}$ and $x \in X$ satisfying $f_i(x) < f_i(\hat{x})$ there exists an index $j \in \{1, 2, ..., p\}$ such that $f_j(x) > f_j(\hat{x})$ and

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq M.$$

Definition

A feasible solution $\hat{x} \in X$ is called properly efficient solution of MOP in the sense of Henig, if $(f(\hat{x}) - C) \cap f(X) = \{f(\hat{x})\}$, for some convex pointed cone C with $\mathbb{R}^p_{\geqq} \backslash \{0\} \subseteq int(C)$.

Theorem

The above two definitions are equivalent for MOP.

Definition

A feasible solution $\hat{x} \in X$ is called properly efficient solution of MOP in the sense of Geoffrion, if it is efficient and there is a real number $M > 0$ such that for all $i \in \{1, 2, ..., p\}$ and $x \in X$ satisfying $f_i(x) < f_i(\hat{x})$ there exists an index $j \in \{1, 2, ..., p\}$ such that $f_j(x) > f_j(\hat{x})$ and

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Theorem

The above two definitions are equivalent for MOP.

Benson definition, Borwein definition, etc.

¶ For $\Omega \subseteq \mathbb{R}^n$ and $x \in cl\Omega$, the tangent cone of Ω at x, denoted by $T_\Omega(\bar{x})$, is defined by

$$T_\Omega(\bar{x}) = \left\{ d \in \mathbb{R}^n \ : \ \exists\big(\{x_i\} \subseteq \Omega, \ \{t_i\} \subseteq \mathbb{R}\big); \ t_i \downarrow 0, \ \frac{x_i - \bar{x}}{t_i} \longrightarrow d \right\}.$$

Definition

Let $h : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz near a given point $x \in \mathbb{R}^n$. The generalized directional derivative of h at x in the direction v, denoted by $h^\circ(x; v)$, is defined as $h^\circ(x; v) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{h(y + tv) - h(y)}{t}$.

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Definition

Clarke (2013) A function $h : \mathbb{R}^n \to \mathbb{R}$ is called regular at $\bar{x}$ if $h^\circ(\bar{x}; d)$ exists and $h^\circ(\bar{x}; d) = \lim_{t \downarrow 0} \frac{h(\bar{x}+td) - h(\bar{x})}{t}$ for each $d \in \mathbb{R}^n$.

Definition

Let $\bar{x} \in X$ be an efficient solution of (MOP). $\bar{x}$ is called a robust efficient solution if there exists $\epsilon > 0$ such that for any $p \times n$ matrix C with $\|C\| < \epsilon$, the vector $\bar{x}$ is an efficient solution to

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The converse of the above theorem does not hold necessarily, even for the linear case.

Example

[Compactness is essential]:

$$\min\ (-x, x^3) \quad s.t.\ \ x \in \mathbb{R}.$$

It is not difficult to see that $\bar{x} = 1$ is a robust efficient solution (consider $\epsilon = 0.1$), while the problem does not have any properly efficient solution.

Definition

$d \in \mathbb{R}^n$ is called a non-ascent direction of f at $\bar{x}$ (denoted by $d \in G(\bar{x})$) if $d^T\xi \leq 0$, for each $\xi \in \partial f_i(\bar{x})$ and each $i \in \{1, 2, \ldots, p\}$.

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Theorem

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The condition given in the above theorem is not sufficient for robustness in general case.

Example

Consider the MOP:

$$\begin{aligned} &\min \ (f_1(x), f_2(x)) \\ &s.t. \ x \in \mathbb{R}, \end{aligned}$$

in which

$$f_1(x) := x, \qquad f_2(x) := \begin{cases} -x & |x| < 1, \\ -x^{(\frac{1}{3})} & |x| \geq 1. \end{cases}$$

Let $\bar{x} = 2$. We have $T_X(\bar{x}) = \mathbb{R}$ and $G(\bar{x}) = \{0\}$.
$\bar{x} = 2$ is an efficient solution to the above problem, while for any $\epsilon > 0$ it is not an efficient solution to

$$\begin{aligned} &\min \ (f_1(x), f_2(x) + \tfrac{\epsilon}{2}x) \\ &s.t. \ \ x \in \mathbb{R}, \end{aligned}$$

because for each $\epsilon > 0$, by setting $x_\epsilon = \min\{-125, \frac{-1}{\epsilon^3}\}$, we have $f_1(x_\epsilon) < f_1(2)$ and $f_2(x_\epsilon) \leq f_2(2)$.

Theorem

Let X be a closed and convex set and f_i $(i = 1, \ldots, p)$ be convex. Assume that $\bar{x}$ is an efficient solution to (MOP). $\bar{x}$ is a robust efficient solution to (MOP) if and only if $T_X(\bar{x}) \cap G(\bar{x}) = \{0\}$.

Consider

$$(MOP-g): \qquad \min\{f(x) \;\; s.t. \;\; g_i(x) \leq 0, \quad i=1,2,...,m\},$$

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$\bar{x}$ is robust & (CQ)

$$\Longrightarrow P_{\bar{x}} := Pos\left(\bigcup_{i=1}^{p} \partial f_i(\bar{x}) \right) + Pos\left(\bigcup_{i \in A(\bar{x})} \partial g_i(\bar{x}) \right) = \mathbb{R}^n.$$

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Theorem

Let f_i, g_j functions be convex. If $\bar{x}$ is an efficient solution and $P_{\bar{x}} = \mathbb{R}^n$, then $\bar{x}$ is a robust efficient solution to Problem (MOP-g).

Theorem

Let f_i $(i = 1, 2, \ldots, p)$ and g_j $(j = 1, 2, \ldots, m)$ be convex in Problem (MOP-g). If $\bar{x}$ is a robust efficient solution of Problem (MOP-g) which satisfies (CQ), then $\bar{x}$ is a proper efficient solution of (MOP-g).

Robustness radius:

Lemma

Let X be a closed and convex set and f_i $(i = 1, ..., p)$ be convex. Let $d \in T_X(\bar{x})$ with $\|d\| = 1$. If $\bar{x}$ is a robust solution of (MOP), then $\|(f'(\bar{x}; d))^+\| > 0$ and it is equal to the optimal value of the following problem,

$$\sup\{t : f'(\bar{x}; d) + tCd \notin -\mathbb{R}^p_+, \ \ \forall \|C\| \leq 1\}.$$

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Theorem

Under the assumptions of the above lemma, the optimal value of the following problem is positive and it is a robustness radius for $\bar{x}$.

$$\begin{aligned} \min\ & \|(f'(\bar{x}; d))^+\| \\ s.t.\ & d \in T_X(\bar{x}), \\ & \|d\| = 1 \end{aligned}$$

Comparison with worst case-based notions:

Let U be an uncertain set and $X \subseteq \mathbb{R}^n$ be the set of feasible solutions. Also, let $f_i : X \times U \longrightarrow \mathbb{R}$ for $i = 1, 2, \ldots, p$ be objective functions. For a feasible decision variable vector $x \in X$ and a $u \in U$, the value of objective function is denoted by $f(x, u)$. Define $F : X \to \mathbb{R}^p$ by

$$F_i(x) = \max_{u \in U} f_i(x, u), \quad i = 1, 2, \ldots, p.$$

A feasible vector $\bar{x} \in X$ is called a robust solution in the sense of Fliege and Werner (FW) if it is an efficient solution to the following multi-objective problem

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$$\min F(x)$$
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Theorem

Let $\bar{x}$ be a robust solution of (MOP), in the sense of norm, with radius ϵ. Then considering any $\bar{\epsilon} \in (0, \epsilon)$, the vector $\bar{x}$ is a robust solution in the sense of FW with $U = \{C_{p \times n} : \|C^i\| \leq \frac{\bar{\epsilon}}{\sqrt{p}}, \quad \forall i = 1, 2, \ldots, p\}$ and $f(x, C) = f(x) + Cx$.

Comparison with set-based robustness:

Ehrgott et al. (2014) (EIS in brief) have defined a feasible point $\bar{x} \in X$ as a robust solution if there is no $x \in X$ such that

$$f_U(x) \subseteq f_U(\bar{x}) - (\mathbb{R}^p_{\geq} \setminus \{0\})$$

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Theorem

Let $\bar{x}$ be a robust solution of (MOP) in the sense of norm with radius ϵ. Then $\bar{x}$ is a robust solution in the sense of EIS with $f_U(x) = \{f(x) + Cx : \|C\| \leq 0.5\epsilon\}$.

References [Part 2]

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THANKS FOR YOUR ATTENTION!


The Second Workshop on
Nonsmooth Optimization and its Applications
Applied Mathematics Group, Urmia University of Technology,
Orūmīyeh, West Azerbaijan, Iran
September 14-15, 2021 (Shahrivar 23-24, 1400)
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Boris Mordukhovich — Wayne State University, The United States
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